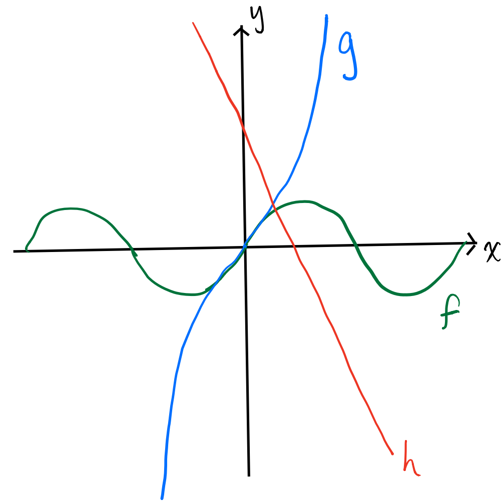
Math 1A: Calculus | **Instructor: Alexander Paulin**

Quiz 1: Fundamentals of Functions

Your name and SID: | *Discussions 201, 203 // 2018-08-31*

Problem 1 (5 points)**.** From the list of choices below, identify the equations corresponding to f, g, and h. You do not need to explain your answers.



- $y = \cos x$
- $y = \sin x$ is the equation of f
- $y = \tan x$ is not the equation of g. I intentionally included this as a misleading answer choice, but remember that $\tan(x)$ is periodic!
- $y = x^3 + x$ is the equation of g
- $y = x^3 + x^2$
- $y = x + 4$
- $y = -2x + 4$ is the equation of h

Problem 2 (5 points)**.** Find the domain of

$$f(x) = \frac{1}{x^2 - |x|} + \sqrt{4x^2 - 1}$$

where $|x|$ denotes the absolute value of x. Is $f(x)$ even, odd, or neither?

Solution: This solution may look long, but the idea isn't that complicated: we just look at the various pieces that f is "built out of" and decide when everything makes sense.

$f(x)$ is defined when both $1/(x^2 - |x|)$ and $\sqrt{4x^2 - 1}$ are defined. Let's consider these two pieces separately.

For the first expression, its numerator 1 and denominator $x^2 - |x|$ are defined everywhere, so we just have to worry about when the denominator is equal to 0. This happens when $x^2 = |x|$. The solutions to this are $x = -1, 0, 1$; this is probably most easily seen by drawing a graph. Hence the domain of $1/(x^2 - |x|)$ consists of all real numbers aside from -1, 0, and 1.

Now let's look at $\sqrt{4x^2 - 1}$. The expression $4x^2 - 1$ is defined everywhere, so we just need to see when $4x^2 - 1 \geq 0$. We can rewrite this as $x^2 \geq 1/4$. This is equivalent to $|x| \geq 1/2$.

So for the whole function $f(x)$ to make sense, we must have $|x| \geq 1/2$ *and* $x \neq -1, 0, 1$. There are a number of ways you could express this; e.g. the domain of f is

$$\boxed{(-\infty, -1) \cup (-1, -1/2] \cup [1/2, 1) \cup (1, \infty).}$$

Finally, f is $\boxed{\text{even}}$ because $f(-x) = f(x)$:

$$f(-x) = \frac{1}{(-x)^2 - |-x|} + \sqrt{4(-x)^2 - 1} = \frac{1}{x^2 - |x|} + \sqrt{4x^2 - 1} = f(x).$$

□

(Continued on back of sheet.)

Problem 3 (5 points)**.** Determine coefficients a, b, c so that the graph of the quadratic $f(x) = ax^2 + bx + c$ passes through the points $(0, 12)$, $(2, 0)$, and $(3, 0)$.

Solution one: You could do this problem by solving the system of equations

$$\begin{aligned} c &= 12 \\ 4a + 2b + c &= 0 \\ 9a + 3b + c &= 0 \end{aligned}$$

obtained by plugging the three data points into the quadratic. I won't spell out the details here, but the solution is

$$\boxed{a = 2, b = -10, c = 12.}$$

□

Solution two: Alternatively, you could solve this by observing that, for any value of s, the graph of $y = s(x-2)(x-3)$ passes through $(2, 0)$ and $(3, 0)$ (why?). Then use $f(0) = 12$ to determine that $s = 2$, and so

$$f(x) = 2(x-2)(x-3) = 2x^2 - 10x + 12.$$ □

Only attempt this problem if you have finished all the others. It is not worth any points and it is significantly harder. Determine the *range* of the function

$$f(x) = \sec(x) + 2\cos(x).$$

Hint: It's a good idea to start by drawing a picture. Then, consider when $-\pi/2 < x < \pi/2$. On that interval, $\sec(x)$ and $\cos(x)$ are both positive. The following inequality is true:

$$\left(\sqrt{\sec(x)} - \sqrt{2}\sqrt{\cos(x)}\right)^2 \geq 0.$$

Expand the left-hand side.

Solution: $f(x)$ is periodic with period 2π. It suffices to look at how f behaves in a single period, such as $(-\pi/2, 3\pi/2)$. We'll look at the interval $(-\pi/2, \pi/2)$ first. As suggested in the hint, when x is in this interval we can write

$$\begin{aligned} \left(\sqrt{\sec(x)} - \sqrt{2}\sqrt{\cos(x)}\right)^2 &\geq 0 \\ \sec(x) - 2\sqrt{2}\sqrt{\sec(x)\cos(x)} + 2\cos(x) &\geq 0 \\ \sec(x) - 2\sqrt{2}\sqrt{1} + 2\cos(x) &\geq 0 \\ \sec(x) + 2\cos(x) &\geq 2\sqrt{2}. \end{aligned}$$

This bound is "tight" in the sense that there is a value of x for which equality holds. It's when $x = \pi/4$ (check this!). For x in the interval $(-\pi/2, \pi/2)$, the function $f(x)$ takes values in $[2\sqrt{2}, \infty)$.

For x in the interval $(\pi/2, 3\pi/2)$, the function $f(x)$ takes values in $(-\infty, 2\sqrt{2}]$. (This is because $f(x+\pi) = -f(x)$.) So altogether the range is

$$\boxed{(-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, \infty).}$$

Once we learn how to differentiate, we'll have a more systematic way of answering questions like this one. □